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Electrical Machines

Fundamentals of Electromechanical Energy Conversion

Taylor & Francis

Contents

Preface

The book is intended to serve as a textbook for basic courses on Electrical Machines covering the fundamentals of the electromechanical energy conversion, transformers, classical electrical machines, i.e., DC brush machines, induction machines, wound-field rotor synchronous machines and modern electrical machines, i.e., switched-reluctance machines (SRM) and permanent magnet brushless machines (PMBM). The author breaks the stereotype that basic Electrical Machine course is limited only to transformers, DC brush machines, AC induction machines and wound-field synchronous machines.

In addition to academic research and teaching, the author has been working for over 18 years in the United States high-technology corporative business, being directly involved in solution to real problems as design, simulation, manufacturing and laboratory testing of large variety of electrical machines for energy generation, conversion and utilization. Supervising young engineers the authors is fully aware what the 21st century industry requires from them and how they should be trained to meet the demands of employers. The author wants to leverage his industrial experience to the classroom teaching. The structure of this book is as follows. Chapter 1 gives a brief overview of electromechanical energy conversion which is needed to understand the next chapters and have broad vision of operation of electromechanical energy devices in power trains. Chapter 2 discusses single phase and three-phase transformers used in power electrical engineering. Chapter 3 considers SRMs as the simplest electrical machines. It also shows that such machines must operate in a machine-solid state converter environment. Chapter 4 briefly discusses DC brush (commutator) machines. Nowadays, these machines are gradually replaced by vector-controlled induction and PM brushless motors, so the presented material is limited to fundamentals of standard DC brushless motors while emphasis is given on modern brush PM slotless motors. Chapter 5 deals with armature windings of AC electrical machines with focus on coils construction, connection diagrams, induced EMF, distribution of the magnetomotive force (MMF), effect of magnetic saturation. The reader can familiarize with such terms as winding factors, saturation of magnetic circuit, effect of slotting and effect of higher space harmonics. This material is crucial to understand AC induction, synchronous and PM brushless machines. Chapter 6 on induction machines is the largest chapter in this book, because induction motors are the most common machines in industrial, traction, energy systems and public life applications. This chapter also introduces fundamentals of inverter-fed induction motors. Chapter 7 is devoted to wound-field rotor synchronous machines with focus on turboalternators. This chapter ends with wound-field rotor synchronous motors, which is a bridge to Chapter 8 on PM brushless motors. PM brushless motors are categorized into sine-wave (synchronous) motors and square wave brushless motors. In this chapter the reader can also find fundamentals of inverter-fed PM brushless motors.

Compared to other textbooks on the subject, the presented book has a number of exclusive features:

(a) It includes SRMs and PMBMs, which are nowadays more popular than DC brush machines;
(b) It deals exhaustively with electrical machines without going deep into mathematical analysis;
(c) It shows examples of application of electrical machines to modern electromechanical drive systems;
(d) It shows technical problems and their possible solutions.

Most of the material of this textbook has been classroom-tested with successful results.

What are the key benefits of the textbook for the reader? Why should they purchase this book?

- The material is presented in as simple way as possible.
- In addition to standard electrical machines, students can familiarize themselves also with modern electrical machines;
- The book is written by the author with both academic and industrial experience;
- Each chapter ends with a Summary containg most important knowledge presented in the given chapter and numerical problems for solution by students, while similar numerical examples are solved in the relevant chapter;
- The book contains over 100 carefully selected numerical examples attractive for students and encouraging them to study;
- The book highlights the elements of power electronics used in systems with electrical machines.

Students using this textbook should have taken courses in circuit theory and electromagnetic field theory as prerequisites. For a one semester course and three lectures per week the author recommend all eight chapters. For two lectures per week the author recommend chapters 2, 4 to 8. A solution manual for lecturers can be obtained from Taylor & Francis, CRC Press.

The author has produced this textbook without any support from funding agencies and/or industry both in European Union countries and the United States of America.

Any suggestions for improvement, constructive criticism and corrections both from students and professors are most welcome.

Glastonbury, CT, January, 2016

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1

INTRODUCTION TO ELECTROMECHANICAL ENERGY CONVERSION

1.1 What is electromechanical energy conversion?

Electromechanical energy conversion is a conversion of mechanical energy into electrical energy (generator) or vice versa (motor) with the aid of rotary motion (rotary machines) or translatory (linear) motion (linear machines and actuators).

Electrical machines, solenoid actuators and electromagnets are generally called *electromechanical energy conversion devices* (Fig. 1.1).

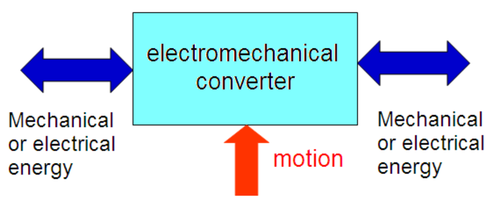

Fig. 1.1. Electromechanical energy conversion.

Transformers and *solid-state converters* do not belong to the group of electromechanical energy conversion devices because they only convert one kind of electrical energy into another kind of electrical energy with different parameters (change in voltage, current, frequency, number of phases, conversion of DC into AC current, etc.) without any motion.

1.1.1 Block diagrams of electromechanical energy conversion devices

Fig. 1.2a shows a block diagram of a motor, while Fig. 1.2a shows a block diagram of a generator. An example of application of an electric motor is shown in Fig. 1.3. An example of application of an electric generator is shown in Fig. 1.4.

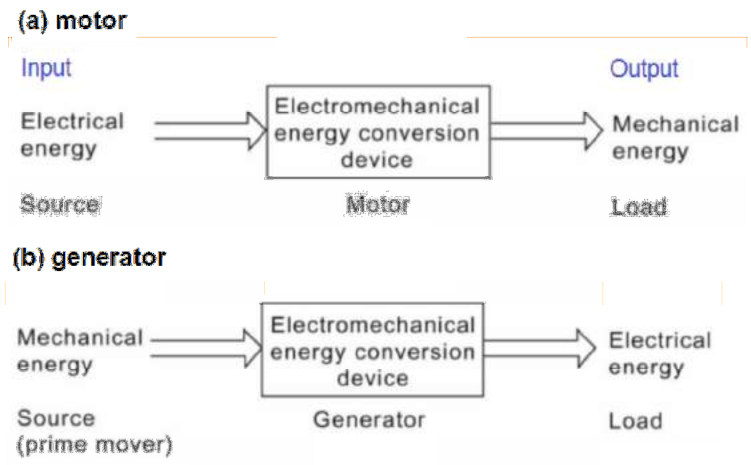

Fig. 1.2. Block diagrams of electromechanical energy conversion devices: (a) motor; (b) generator.

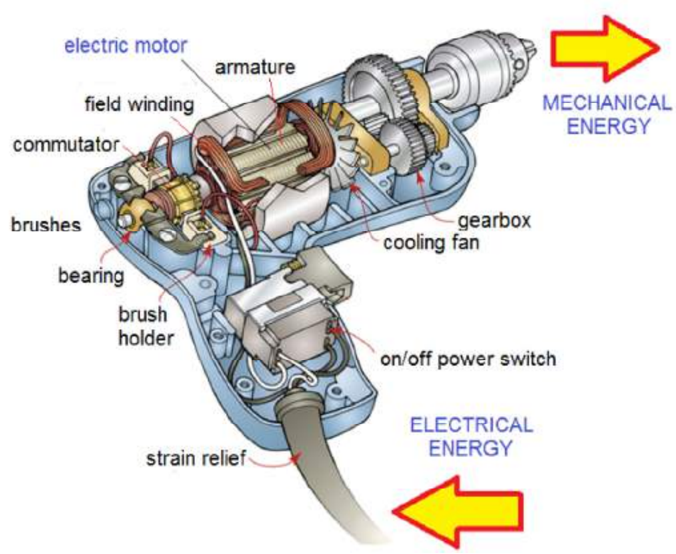

Fig. 1.3. Power tool: an example of conversion of electrical energy into mechanical energy.

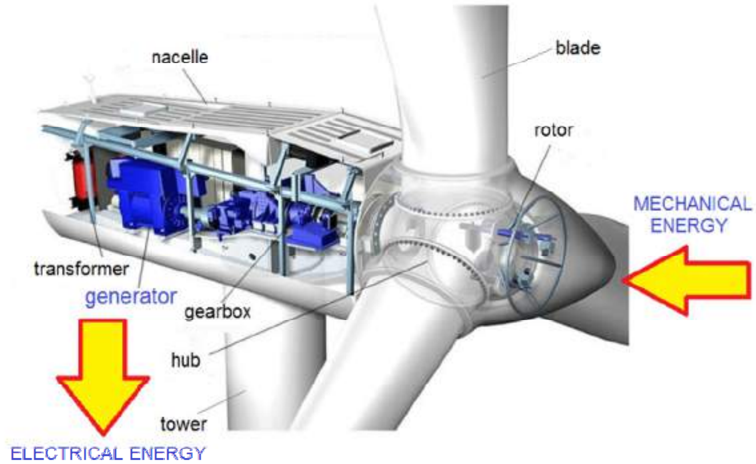

Fig. 1.4. Wind turbine generator: an example of conversion of mechanical energy into electrical energy.

1.1.2 Left-hand and right-hand rule

The left-hand rule (Fig. 1.5a) indicates the direction of the phasor of the electrodynamic force (EDF), i.e.,

$$dF = I d\mathbf{l} \times \mathbf{B} \tag{1.1}$$

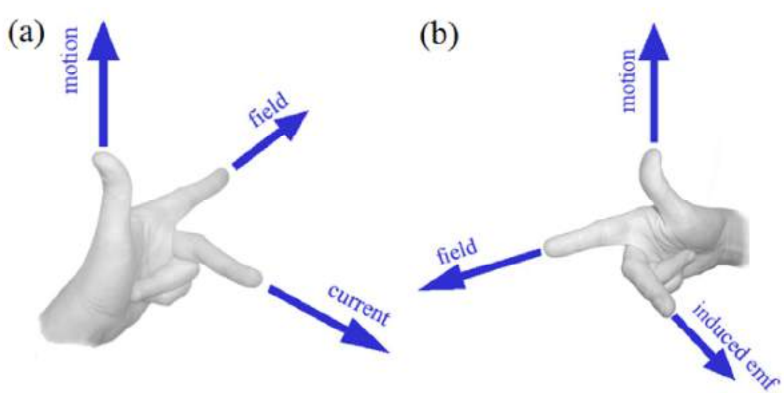

Fig. 1.5. Left-hand and right-hand rules: (a) left-hand rule shows the direction of electrodynamic force (EDF); (b) right-hand rule shows the direction of electromotive force (EMF).

or, in scalar form

$$F = BIl \tag{1.2}$$

The right-hand rule (Fig. 1.5b) indicates the direction of the phasor of the electromotive force (EMF), i.e.,

$$d\mathbf{E} = \mathbf{v} \times \mathbf{B}\ d\mathbf{l} \tag{1.3}$$

or, in scalar form

$$E = Blv \tag{1.4}$$

1.1.3 Energy flow in an electromechanical energy conversion device

Fig. 1.6 illustrates the conversion of electrical energy into mechanical energy, according to the following equation:

$$dW_e = dW_f + dW_{loss} + dW_{mech} \tag{1.5}$$

where dW_e is the electrical energy (input energy) dW_f is the energy stored in the magnetic field (coil), dW_{loss} are all the power losses and dW_{mech} is the mechanical energy (output energy).

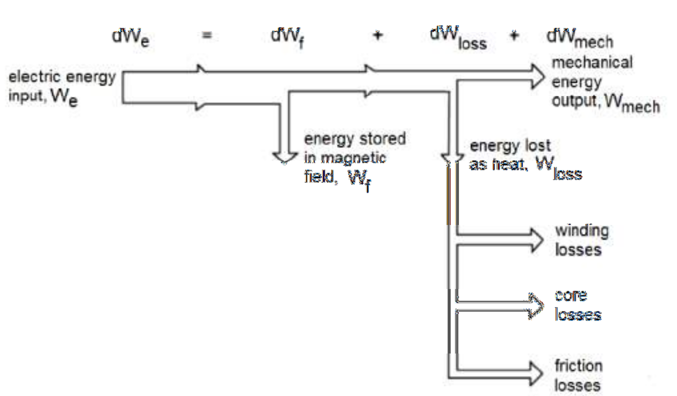

Fig. 1.6. Energy flow in electromechanical energy conversion device.

1.2 Anologies between electric and magnetic circuits

Table 1.1 contains analogies in electric and magnetic circuits, while Table 1.2 compares fundamental equations and laws for electric and magnetic circuits.

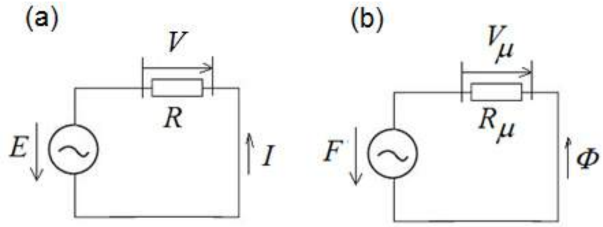

Fig. 1.7. Simple circuits: (a) electric circuit; (b) equivalent magnetic circuit.

Table 1.1. Analogies in electric and magnetic circuits.

Quantity	Electric circuit	Magnetic circuit
Voltage	Electric voltage, V [V]	Magnetic voltage drop, V_μ [A]
Source voltage	Electromotive force (EMF), E [V]	Magnetomotive force (EMF), F [A]
Current/flux	Electric current, I [A]	Magnetic flux, Φ [Wb]
Resistance /reluctance	Resistance R [Ω = [1/S]	Reluctance R_μ [1/H]
Constant	Electric conductivity, σ [S/m]	Magnetic permeability, μ [H/m]

Table 1.2. Comparison of fundamental laws for electric and magnetic circuits.

Law	Electric circuit	Magnetic circuit
Ohm's law	Resistance $R = \frac{V}{I}$ Conductance $G = \frac{I}{V}$	Reluctance $R_\mu = \frac{V_\mu}{\Phi}$ Permeance $\Lambda_\mu = \frac{\Phi}{V_\mu}$
2nd Ohm's law	Resistance $R = \frac{l}{\sigma S}$ Conductance $G = \frac{S}{\rho l}$	Reluctance $R_\mu = \frac{l}{\mu S}$ Permeance $\Lambda_\mu = \frac{\mu S}{l}$
Kirchhoff's current law	Sum of currents $\sum I = 0$	Sum of magnetic fluxes $\sum \Phi = 0$
Kirchhoff's voltage law	Sum of voltage drops $\sum V - \sum RI = 0$	Sum of magnetic voltage drops $\sum V_\mu - \sum R_\mu \Phi = 0$
Faraday's law/Ampere's law	EMF $E = \pi\sqrt{2} f N \Phi$	MMF $F = NI$

1.3 Losses in ferromagnetic cores

The *magnetization curve* $B(H)$ of isotropic silicon cold-rolled steel Armco-DI-Max M19, which is the most popular ferromagnetic material in construction of electrical machines, is plotted in Fig. 1.8. The *specific losses curve* $\delta p(B)$ of the same silicon steel at 50 Hz is plotted in Fig. 1.9. An addition of silicon improves the magnetic properties of steel, i.e., increases the saturation magnetic flux density and reduces the losses.

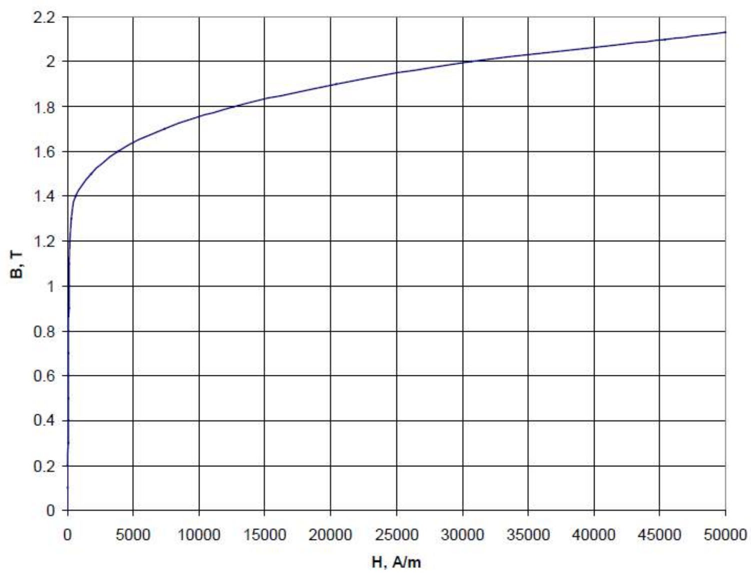

Fig. 1.8. Magnetization curve $B(H)$ of isotropic silicon cold-rolled steel Armco-DI-Max M19.

As the alternating magnetic flux magnetizes the core, the energy is lost in the core due to the hysteresis effect. The energy loss, called the hysteresis loss, is proportional to the area of the hysteresis loop (Fig. 1.10). The hysteresis loss depends on the ferromagnetic material of the core. The first empirical formula for hysteresis losses was proposed by C.P. Steinmetz and published in the 1892 [36], i.e.,

$$\Delta P_h = k_h B_m^n \tag{1.6}$$

where k_h and n are curve fitted coefficients of actual experimental data and B_m is the peak value of the magnetic flux density. A more accurate empirical formula for the hysteresis loss contains the frequency f of the magnetic flux density, i.e.,

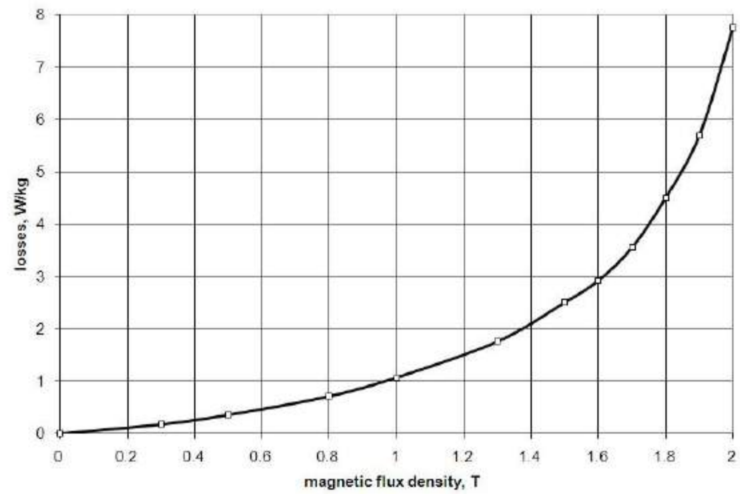

Fig. 1.9. Specific core loss curve $\delta p(B)$ of isotropic silicon cold-rolled steel Armco-DI-Max M19 at 50 Hz.

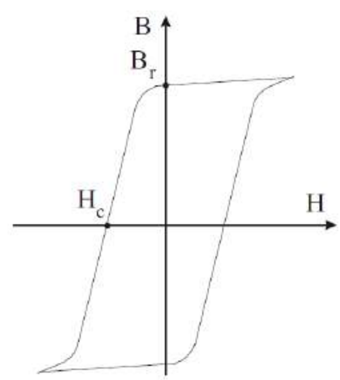

Fig. 1.10. Hysteresis loop. B_r – remanent magnetic flux density, H_c – coercivity.

$$\Delta P_h = c_h f B_m^n \tag{1.7}$$

where B_m is the peak value of the magnetic flux density, f is the frequency and the hysteresis constants c_h and n vary with the core material. The constant n is often assumed to be $1.6 \ldots 2.0$.

Another source of the power loss in the ferromagnetic core is the eddy currents induced by the alternating magnetic flux. If the magnetic flux is perpendicular, directed toward the plane of this page and increasing with the time, it induces voltages in conductive material of the core (Fig. 1.11). Under action of these voltages, eddy-currents flow in closed loops (paths) producing power losses i^2R, which are converted into heat. The eddy-current losses can

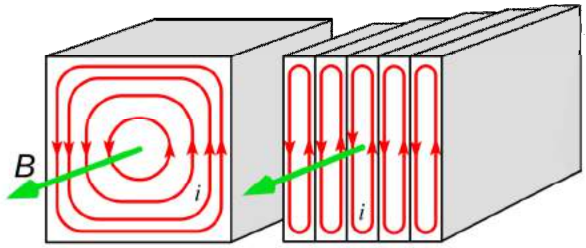

Fig. 1.11. Eddy currents in: (a) solid ferromagnetic core; (b) laminated ferromagnetic core.

be reduced by decreasing the current i or increasing the resistance R. This can be done by replacing a solid ferromagnetic core with laminated ferromagnetic core. The eddy current losses are proportional to the frequency f square and the peak magnetic flux density B_m square, i.e.,

$$\Delta P_e = c_e f^2 B_m^2 \tag{1.8}$$

The eddy-current constant c_e depends on the electric conductivity of the material of the core and the thickness square of laminations. An addition of silicon reduces the electric conductivity of steel, i.e., reduces the eddy-current losses and increases the saturation magnetic flux density.

There are also the so-called *excess eddy current losses*, which can be estimated as [2]

$$\Delta P_{ex} = 8\sqrt{\sigma_{Fe} G S_{Fe} V_0} B^{1.5} f^{1.5} \tag{1.9}$$

where σ_{Fe} is the electric conductivity of steel sheet, $G = 0.1356$ is a unitless constant, V_0 is the curve fitted coefficient and S_{Fe} is a cross-sectional area of the core.

The total power losses, neglecting the excess eddy current loss, are:

$$\Delta P_{Fe} = \Delta P_h + \Delta P_e = c_{Fe} f^{1.3} B_m^2 \tag{1.10}$$

In practical calculations of AC magnetic circuits the core losses ΔP_{Fe} can be estimated on the basis of the specific core losses $\Delta p_{1/50}$ and masses, say, of legs and yokes of a transformer, i.e.,

$$\Delta P_{Fe} = \Delta p_{1/50} \left(\frac{f}{50}\right)^{4/3} \left(k_{adl} B_{ml}^2 m_l + k_{ady} B_{my}^2 m_y\right) \tag{1.11}$$

where $k_{adl} > 1$ and $k_{ady} > 1$ are the factors accounting for the increase in losses due to metallurgical and manufacturing processes, $\Delta p_{1/50}$ is the specific core loss in W/kg at 1 T and 50 Hz, B_l is the magnetic flux density in the leg, B_y is the magnetic flux density in the core (yoke), m_l is the mass of legs, and m_y is the mass of yokes.

Table 1.3. Magnetization and specific core loss characteristics of three types of cold-rolled, nonoriented electrotechnical silicon steel sheets, i.e., Dk66, thickness 0.5 mm, $k_i = 0.96$ (Sweden); H-9, thickness 0.35 mm, $k_i = 0.96$ (Japan); and DI-MAX EST20, thickness 0.2 mm, $k_i = 0.94$, Italy).

B	H, A/m			Specific core losses Δp, W/kg				
T	Dk66	H9	DI-MAX EST20	Dk66	H9		DI-MAX EST 20	
				50 Hz	50 Hz	60 Hz	50 Hz	400 Hz
0.1	55	13	19	0.15	0.02	0.02	0.08	0.30
0.2	65	20	28	0.24	0.06	0.10	0.15	0.70
0.4	85	30	37	0.50	0.15	0.20	0.25	2.40
0.6	110	40	48	0.90	0.35	0.45	0.42	6.00
0.8	135	55	62	1.55	0.60	0.75	0.63	
1.0	165	80	86	2.40	0.90	1.10	0.85	
1.2	220	160	152	3.30	1.30	1.65	1.25	
1.4	400	500	450	4.25	1.95	2.45	1.70	
1.5	700	1500	900	4.90	2.30	2.85	1.95	
1.6	1300	4000	2400		2.65	3.35	2.20	
1.7	4000	6500	6500					
1.8	8000	10,000	17,000					
1.9	15,000	16,000						
2.0	22,500	24,000						
2.1	35,000							

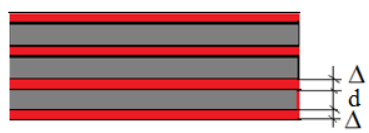

Fig. 1.12. Stacking factor.

To reduce eddy-current losses in sheet steels, all electrical steels are coated double-sided with a thin layer of insulation, usually oxide insulation. The *stacking factor* is the ratio of thickness of bare sheet to the thickness of sheet with insulation (Fig. 1.12), i.e.,

$$k_i = \frac{d}{d + 2\Delta_i} \tag{1.12}$$

where d is the thickness of bare sheet and Δ_i is the thickness of one-sided insulation.

1.4 Inductor

1.4.1 Ideal inductor

An ideal inductor (Fig. 1.13) has:

- The resistance of the coil $R = 0$;
- Infinitely large magnetic permeability of its core;
- No core losses;
- No leakage magnetic flux, which means that the whole magnetic flux is within the ferromagnetic core and coupled wit the coil.

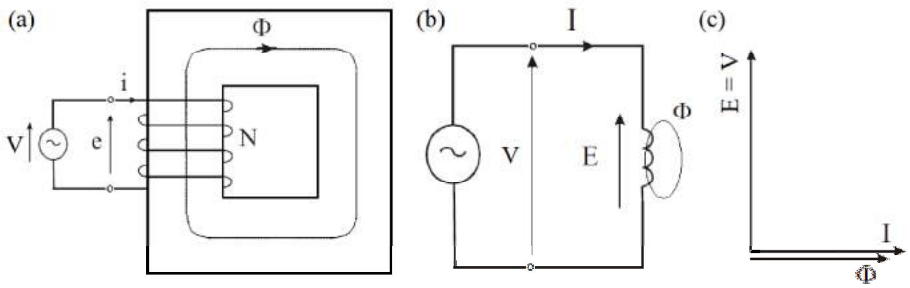

Fig. 1.13. Ideal inductor: (a) coil with zero resistance wound on ferromagnetic core without losses; (b) equivalent circuit; (c) phasor diagram.

Assuming a linear relationship between the current and magnetic flux, the sinusoidal current, i.e.,

$$i = I_m \sin(\omega t) \tag{1.13}$$

produces a sinusoidal magnetic flux

$$\Phi = \Phi_m \sin(\omega t) \tag{1.14}$$

The voltage induced in N-turn coil is

$$e = N\frac{d\Phi}{dt} = N\omega\Phi_m \cos(\omega t) = E_m \cos(\omega t) \tag{1.15}$$

where $E_m = N\omega\Phi_m$. The *rms* voltage is

$$E = \frac{E_m}{\sqrt{2}} = \frac{N\omega\Phi_m}{\sqrt{2}} = \frac{2\pi}{\sqrt{2}}NF\Phi_m = \pi\sqrt{2}Nf\Phi_m \tag{1.16}$$

In eqn (1.16) $\pi\sqrt{2} \approx 4.44 = 4 \times 1.11$ where 1.11 is the form factor for electric and magnetic quantities sinusoidally varying with time. The EMF expressed in terms of the inductance L and current i in the coil

$$e = L\frac{di}{dt} \tag{1.17}$$

For a sinusoidal current

$$e = \omega L I_m \cos(\omega t) = E_m \cos(\omega t) \tag{1.18}$$

The effective value of the EMF expressed in complex form is

$$\mathbf{E} = jX_m\mathbf{I} \tag{1.19}$$

where $X_m = \omega L$ is the magnetizing reactance. For an ideal inductor, the induced voltage (EMF, E) is equal to the supply voltage $E = V$. The equivalent circuit and phasor diagram of an ideal inductor is shown in Fig. 1.13.

1.4.2 Practical inductor

A practical inductor has a real coil with its resistance R and a real ferromagnetic core, in which the hysteresis losses (1.7) and eddy-current losses (1.8) are produced.

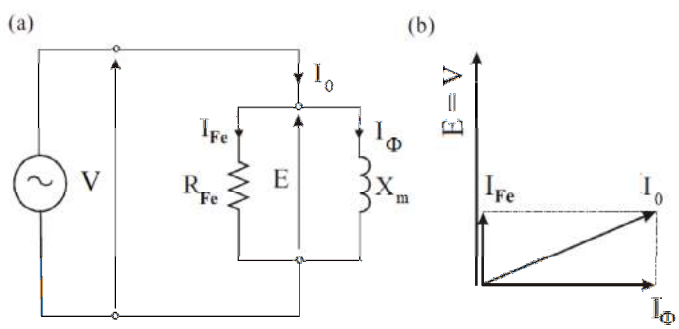

Fig. 1.14. Inductor with core losses: (a) equivalent circuit; (b) phasor diagram.

To draw the equivalent circuit of a practical inductor, an equivalent resistance R_{Fe}, in parallel with X_m, representing the core losses ΔP_{Fe} is introduced. Thus, the total power losses in the core at constant frequency can be expressed as

$$\Delta P_{Fe} = \frac{E^2}{R_{Fe}} \tag{1.20}$$

The power losses ΔP_{Fe} are proportional to the EMF E square, which appears across the resistance R_{Fe} that in turn represents the core losses. This allows building the equivalent circuit of a practical inductor (Fig. 1.14). The exciting current I_0 is split into two components:

- The magnetizing current I_{Φ};
- The core loss current I_{Fe} proportional to the core power losses.

The currents I_{Φ} and I_{Fe} are displaced from each other by an angle $\pi/2$, i.e.,

$$\mathbf{I}_0 = I_{Fe} + jI_{\Phi} \tag{1.21}$$

So far, the resistance R of the coil and the leakage flux Φ_σ have not been taken into account. The leakage flux Φ_σ is the flux, which goes through the air. It induces the voltage e_σ in the coil, which is equal

$$e_\sigma = L_\sigma \frac{di}{dt} = \omega L_\delta \cos(\omega t) \tag{1.22}$$

or in complex notation

$$\mathbf{E}_\sigma = jX_\sigma \mathbf{I} \tag{1.23}$$

where $X_\sigma = \omega L_\sigma$ is the leakage reactance and L_σ is the leakage inductance. The voltage balance equation for the equivalent circuit is

$$\mathbf{V} = \mathbf{E} + R\mathbf{I}_0 + \mathbf{E}_\sigma = \mathbf{E} + R\mathbf{I}_0 + jX_\sigma\mathbf{I}_0 = \mathbf{E} + (R + jX_\sigma)\mathbf{I}_0 \tag{1.24}$$

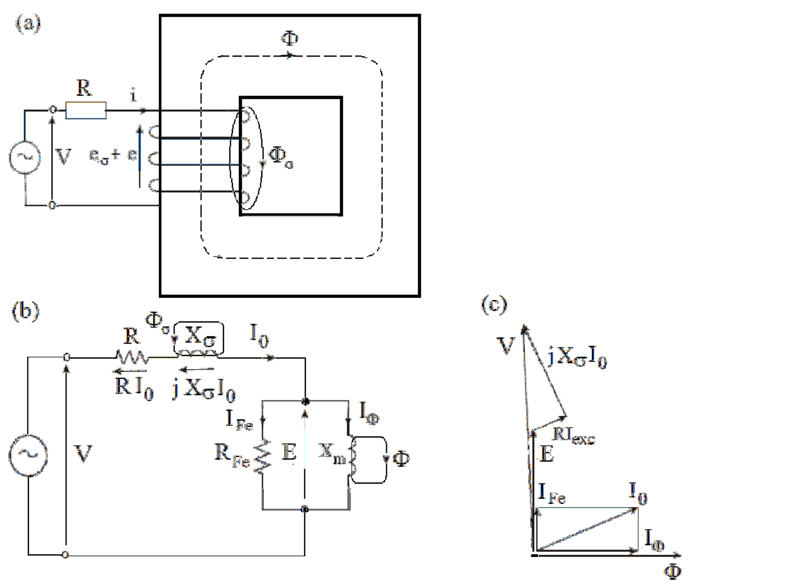

Fig. 1.15. Practical inductor: (a) coil with $R \neq 0$ wound on ferromagnetic core with losses and producing leakage flux; (b) equivalent circuit; (c) phasor diagram.

The equivalent circuit and phasor diagram of a practical inductor with $R \neq 0$, core losses and leakage flux taken into account is sketched in Fig. 1.15.

Example 1.1

Fig. 1.16 shows a magnetic circuit with air gap g. Two series connected windings are placed on external legs and fed with DC voltage $V = 48$ V. Such magnetic circuit can be used, for example, in magnetizers. The cross-section of laminated ferromagnertic core is $25 \times 20 = 500$ mm^2. The mean length of the magnetic flux path in each external section is $l_{Fe1} = l_{Fe2} = 170$ mm. The length of the magnetic flux path in the central leg is $l_{Fe} = 64$ mm. For 0.5-mm thick cold-rolled electrical sheet steel the stacking factor $k_i = 0.96$ and the relative magnetic permeablity $\mu_r = 1000$ (this is a rough simplification, why?).

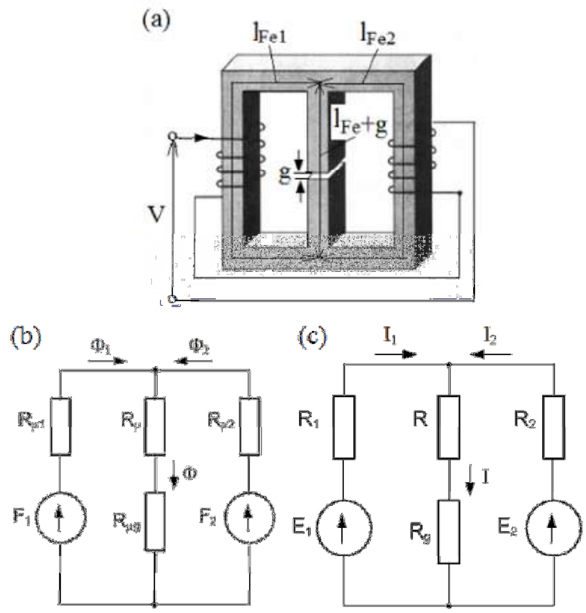

Fig. 1.16. Magnetic circuit with air gap: (a) 3D image; (b) equivalent magnetic circuit with concentrated parameters; (c) equivalent electric circuit with concentrated parameters.

The air gap $g = 6$ mm. Each of the two windings consists of $N = 2000$ turns wound with copper wire with its diameter $d = 0.5$ mm (without insulation). Find magnetic fluxes in each column and the air gap.

Solution

For the equivalent electric circuit (Fig. 1.16c), the following Kirchhoffs's equations can be written:

$$E_1 - I_1 R_1 - I(R + R_g) = 0 \tag{1.25}$$

$$E_2 - I_2 R_2 - I(R + R_g) = 0 \tag{1.26}$$

$$I = I_1 + I_2 \tag{1.27}$$

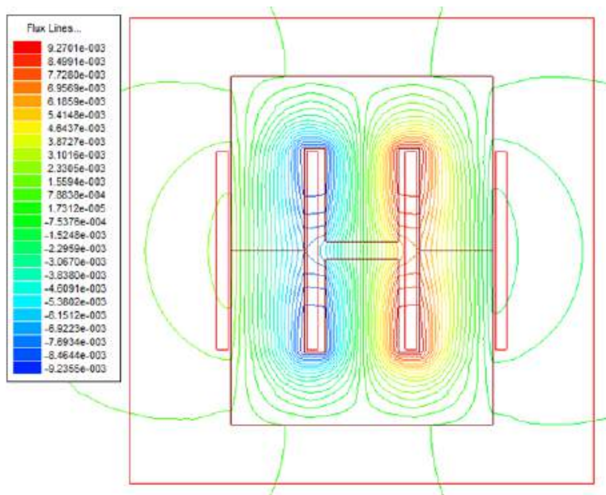

Fig. 1.17. Magnetic flux lines as obtained from the 2D FEM.

Solving the set of the above equations, the currents I_1, I_2 and I are

$$I_1 = \frac{E_1(R_2 + R + R_g) - E_2(R + R_g)}{R_1 + R + R_g) - (R + R_g)^2} \tag{1.28}$$

$$I_2 = \frac{E_2(R_2 + R + R_g) - E_2(R + R_g)}{R_1 + R + R_g) - (R + R_g)^2} \tag{1.29}$$

$$I_2 = \frac{E_1 R_2 + E_2 R_1}{R_1 + R + R_g) - (R + R_g)^2} \tag{1.30}$$

Replacing in eqns (1.28), (1.29) and (1.30) the EMF with MMF and resistances with reluctances, the magnetic fluxes are

$$\Phi_1 = \frac{F_1(R_{\mu 2} + R_\mu + R_{\mu g}) - F_2(R_\mu + R_{\mu g})}{R_{\mu 1} + R_\mu + R_{\mu g}) - (R_\mu + R_{\mu g})^2} \tag{1.31}$$

$$\Phi_2 = \frac{F_2(R_{\mu 2} + R_\mu + R_{\mu g}) - F_1(R_\mu + R_{\mu g})}{R_{\mu 1} + R_\mu + R_{\mu g}) - (R_\mu + R_{\mu g})^2} \tag{1.32}$$

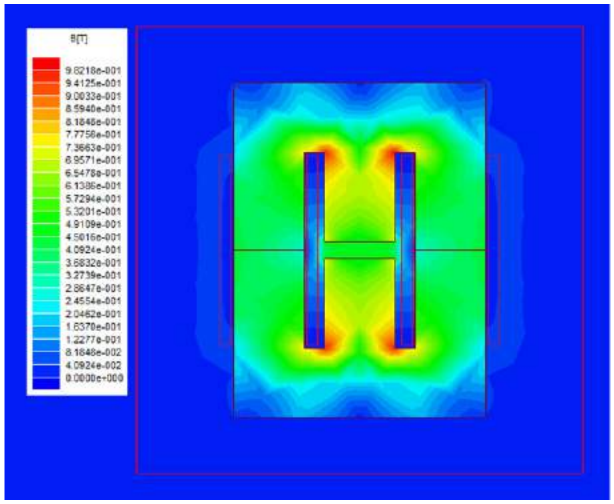

Fig. 1.18. Magnetic flux density distribution as obtained from the 2D FEM. The air gap magnetic flux density $B_g = 0.47$ T obtained analytically is almost the same as that obtained using the FEM (green color).

$$\Phi = \frac{F_1 R_{\mu 2} - F_2 R_{\mu 1}}{(R_{\mu 1} + R_{\mu} + R_{\mu g}) - (R_{\mu} + R_{\mu g})^2} \tag{1.33}$$

The values of reluctances have been calculated as follows:

$$R_{\mu 1} = \frac{l_{Fe1}}{\mu_0 \mu_r ab} = \frac{0.17}{0.4\pi \times 10^{-6} \times 1000 \times 0.25 \times 0.20} = 2.706 \times 10^5 \ 1/\text{H}$$

$$R_{\mu 2} = \frac{l_{Fe2}}{\mu_0 \mu_r ab} = \frac{0.17}{0.4\pi \times 10^{-6} \times 1000 \times 0.25 \times 0.20} = 2.706 \times 10^5 \ 1/\text{H}$$

$$R_{\mu} = \frac{l_{Fe}}{\mu_0 \mu_r ab} = \frac{0.064}{0.4\pi \times 10^{-6} \times 1000 \times 0.25 \times 0.20} = 1.019 \times 10^5 \ 1/\text{H}$$

$$R_{\mu g} = \frac{g}{\mu_0 \mu_r ab} = \frac{0.006}{0.4\pi \times 10^{-6} \times 1000 \times 0.25 \times 0.20} = 95.49 \times 10^5 \ 1/\text{H}$$

The mean length of turn of the winding

$$l_{mean} = 2(a + \Delta i + b + \Delta i) = 2(0.025 + 0.0015 + 0.020 + 0.0015) = 0.096 \text{ m}$$

where $\Delta i = 1.5$ mm is the assumed thickness of the winding–to–core insulation. The cross-section of the round conductor

$$s_w = \frac{\pi d^2}{4} = \frac{\pi \times 0.0005^2}{4} = 0.2 \times 10^{-6} \text{ m}^2$$

The electric conductivity of copper at 75° (hot winding) is 47×10^6 S/m. The resistance of two windings in series

$$R_w = 2\frac{N_c l_{mean}}{\sigma_{75} s_w} = \frac{2000 \times 0.096}{47 \times 10^6 \times 0.2 \times 10^{-6}} = 41.85\ \Omega$$

The current in the winding and current density

$$I_w = \frac{V}{R_w} = \frac{48.0}{41.85} = 1.175 \text{ A} \qquad J_w = \frac{I_w}{s_w} = \frac{1.175}{0.2 \times 10^{-6}} = 5.85 \times 10^6 \text{ A/m}^2$$

The MMFs of windings

$$F_1 = N_c I_w = 2000 \times 1.175 = 2340 \text{ A} \qquad F_2 = N_c I_w = 2000 \times 1.175 = 2340 \text{ A}$$

Magnetic fluxes are calculated using eqns (1.31), (1.32), and (1.33)

$$\phi_1 = 1.175 \times 10^{-4} \text{ Wb} \qquad \phi_2 = 1.175 \times 10^{-4} \text{ Wb} \qquad \phi = 2.35 \times 10^{-4} \text{ Wb}$$

The air gap magnetic flux density

$$B_g = \frac{\Phi}{ab} = \frac{2.35 \times 10^{-4}}{0.025 \times 0.020} = 0.47 \text{ Wb}$$

The magnetic flux lines and magnetic flux density distribution as obtained from the 2D finite element method (FEM) are plotted in Figs 1.17 and 1.18.

1.5 Two magnetically coupled electric circuits

Fig. 1.19 shows two electrical circuits, which are magnetically coupled. These two circuits can create, for example, a single-phase, two-winding transformer.

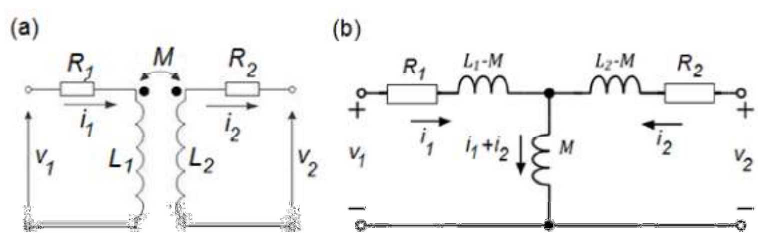

Fig. 1.19. Two magnetically coupled electric circuits: (a) magnetic coupling; (b) T-type electric equivalent circuit.

On the basis of Kirchhoff's voltage law (Fig. 1.19a)

$$v_1 - R_1 i_1 - L_1 \frac{di_1}{dt} - M \frac{di_2}{dt} = 0$$

$$v_2 - R_2 i_2 - L_2 \frac{di_2}{dt} - M \frac{di_1}{dt} = 0 \tag{1.34}$$

where v_1, i_1, R_1, L_1 are the voltage, current, resistance and self-inductance of the primary winding, respectively, v_2, i_2, R_2, L_2 are the same quantities for the secondary winding and M is the mutual inductance between the primary and secondary windings. Eqns (1.34) can be brought to the following form (Fig. 1.19b)

$$v_1 - R_1 i_1 - (L_1 - M) \frac{di_1}{dt} - M \frac{di_1}{dt} - M \frac{di_2}{dt} = 0$$

$$v_2 - R_2 i_2 - (L_2 - M) \frac{di_2}{dt} - M \frac{di_2}{dt} - M \frac{di_1}{dt} = 0 \tag{1.35}$$

Thus, the two magnetically coupled electric circuits can be replaced with the T-type electric equivalent circuit (four-terminal network).

1.6 Doubly-excited rotary device

Fig. 1.20 shows a rotary electromechanical energy conversion device with two electrical inputs (gates). It is shortly called a "doubly-excited device".

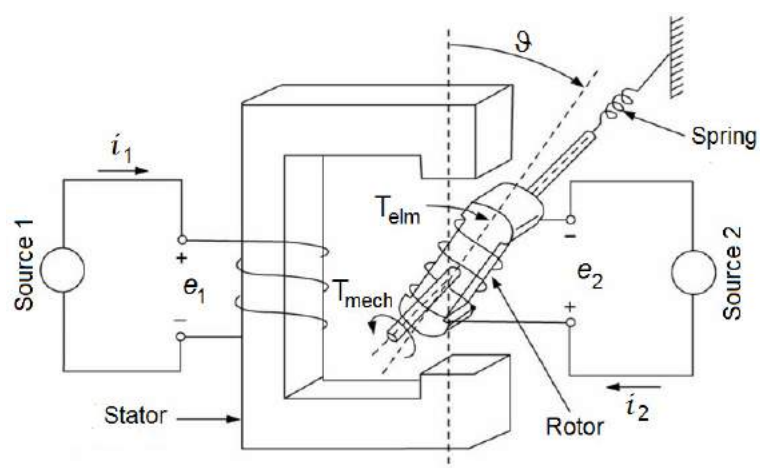

Fig. 1.20. Doubly-excited rotary device.

For a doubly-excited device, the relationships between linkage magnetic fluxes and currents are described by the following equations:

$$\psi_1 = L_{11}(\vartheta)i_1 + L_{12}(\vartheta)i_2$$

$$\psi_2 = L_{21}(\vartheta)i_1 + L_{22}(\vartheta)i_2 \tag{1.36}$$

The self-inductances L_{11}, L_{22} and the mutual inductances $L_{12} = L_{21}$ can be expressed as functions of the angle ϑ, i.e.,

$$L_{11}(\vartheta) = \frac{1}{2}(L_{1d} + L_{1q}) + \frac{1}{2}(L_{1d} - L_{1q}) = L_1 + \Delta L_1 \cos(2\vartheta) \tag{1.37}$$

$$L_{22}(\vartheta) = \frac{1}{2}(L_{2d} + L_{2q}) + \frac{1}{2}(L_{2d} - L_{2q}) = L_2 + \Delta L_2 \cos(2\vartheta) \tag{1.38}$$

$$L_{12}(\vartheta) = L_0 \cos(\vartheta) \tag{1.39}$$

where

$$L_1 = \frac{1}{2}(L_{1d} + L_{1q}) \qquad L_2 = \frac{1}{2}(L_{2d} + L_{2q}) \tag{1.40}$$

$$\Delta L_1 = \frac{1}{2}(L_{1d} - L_{1q}) \qquad \Delta L_2 = \frac{1}{2}(L_{2d} - L_{2q}) \tag{1.41}$$

L_{1d}, L_{2d} are the direct axis (d-axis) inductances, and L_{1q}, L_{2q} are the quadrature axis (q-axis) inductances. The d-axis is the axis of the magnetic flux (center axis of a pole). The q-axis is orthogonal to the d-axis. The following trigonometric identities have been used:

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x) \qquad \cos^2 x = \frac{1}{2}(1 + \cos 2x) \tag{1.42}$$

For $\vartheta = 0$, the mutual inductances $L_{11} = L_d$ and $L_{12} = L_0$. For $\vartheta = \pi/2$, the mutual inductances $L_{11} = L_q$ and $L_{12} = 0$. The inductances $L_{11}(\vartheta)$ and $L_{12}(\vartheta)$ for $L_d = 0.01$ H and $L_q = 0.002$ H have been plotted against the angle ϑ in Fig. 1.21.

The electromagnetic torque can be found, for example, as a sum of first derivatives of inductances with respect to the rotation angle ϑ (magnetic energy stored)

$$T_{elm} = \frac{1}{2}i_1^2\frac{dL_{11}(\vartheta)}{d\vartheta} + \frac{1}{2}i_2^2\frac{dL_{22}(\vartheta)}{d\vartheta} + i_1 i_2\frac{dL_{12}(\vartheta)}{d\vartheta} \tag{1.43}$$

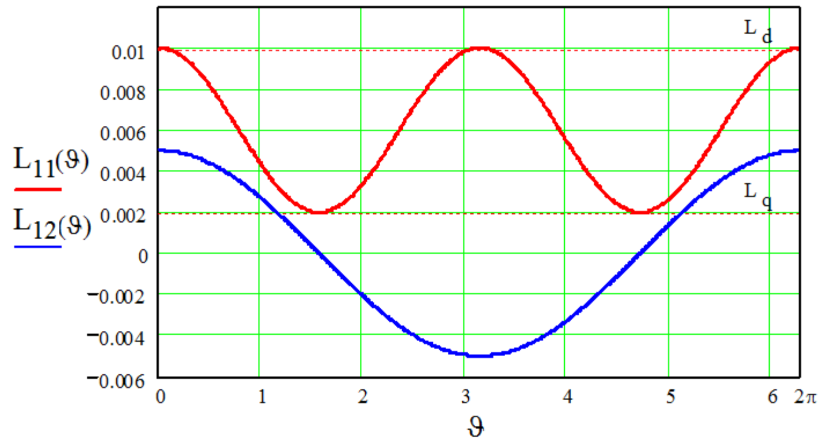

Fig. 1.21. Inductances $L_{11}(\vartheta)$ and $L_{12}(\vartheta)$ for $L_d = 0.01$ H and $L_q = 0.002$H as functions of the angle ϑ.

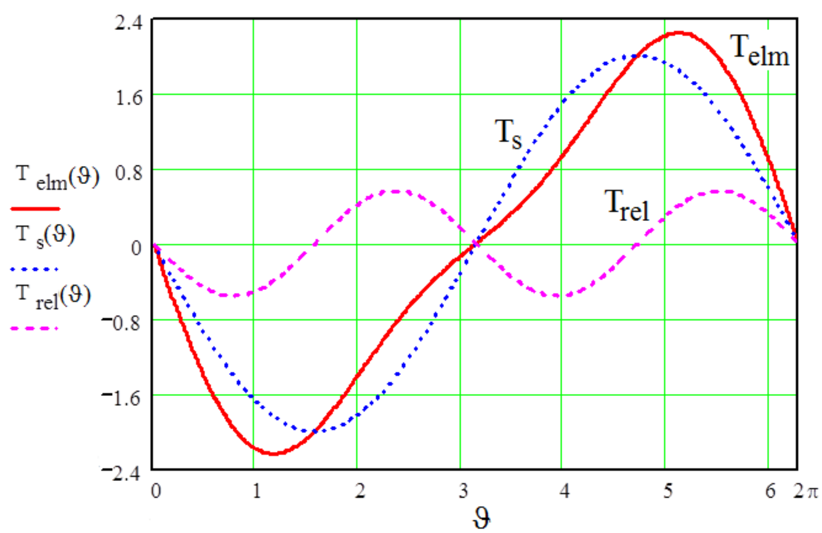

Fig. 1.22. Torques T_{elm}, T_s and T_{rel} as functions of the rotation angle ϑ.

The derivatives of inductances

$$\frac{dL_{11}(\vartheta)}{d\vartheta} = -2\Delta L_1 \sin(2\vartheta)$$

$$\frac{dL_{22}(\vartheta)}{d\vartheta} = -2\Delta L_2 \sin(2\vartheta) \tag{1.44}$$

$$\frac{dL_{12}(\vartheta)}{d\vartheta} = -L_0 \sin(\vartheta)$$

Thus, the electromagnetic torque is

$$T_{elm} = -(i_1^2 \varDelta L_1 + i_2^2 \varDelta L_2) \sin(2\vartheta) - i_1 i_2 L_0 \sin(\vartheta) \tag{1.45}$$

The first term at the right-hand side of eqn (1.45) is the electromagnetic torque produced due to differences in inductances (reluctances) in the direct (d) and quadrature (q) axis. The magnitude of the *reluctance torque* is

$$T_{mrel} = i_1^2 \varDelta L_1 + i_2^2 \varDelta L_2$$

The second term at the right-hand side of eqn (1.45) depends on currents in the stator and rotor windings. The torque is called the *synchronous torque* with the magnitude

$$T_{ms} = i_1 i_2 L_0$$

Thus, the resultant electromagnetic torque can be expressed with the aid of a simple equation

$$T_{elm} = -\left[T_{ms} \sin(\vartheta) + T_{mrel} \sin(2\vartheta)\right]$$

The torques T_{elm}, T_s and T_{rel} are plotted against the rotation angle ϑ in Fig. 1.22. The following values of inductances and currents have been assumed: $L_{1d} = 1.0$ H, $L_{1q} = 0.2$ H, $L_{2d} = 0.5$ H, $L_{2q} = 0.1$ H, $i_1 = 1.0$ A, $i_2 = 2.0$ A.

Example 1.2.

The magnetic circuit shown in Fig. 1.20 is made of high-permeability electrical steel. The rotor does not have any winding and is free to turn about a vertical axis. The leakage flux and fringing effect are neglected.

(a) Derive an expression for the torque acting on the rotor in terms of the dimensions and the magnetic field in the two air gaps. Assume the reluctance of the steel to be negligible, i.e., $\mu \to \infty$ and neglect the effect of fringing.

(b) The maximum flux density in the overlapping portions of the air gaps is to be limited to approximately 1.6 T to avoid excessive saturation of the steel. Calculate the maximum torque for the radius of the rotor $r = 28$ mm, axial length (perpendicular to the page) of the magnetic circuit $L = 20$ mm and the air gap between the rotor and stator $g = 1.5$ mm.

Solution

(a) Derive an expression for the torque

There are two air gaps g in series, so the air gap magnetic field intensity H_g is

$$H_g = \frac{Ni}{2g}$$

Because the permeability of steel is assumed infinite and B_{Fe} must remain finite, $H_{Fe} = B_{Fe}/\mu = 0$ and the coenergy density in the steel is zero, i.e., $(\mu H_{Fe}^2/2 = B_{Fe}^2/(2\mu) = 0)$. Hence the system coenergy is equal to that of the air gaps, in which the coenergy density in the air gap is $\mu_0 H_g^2/2$. The volume of the two overlaping air gaps is $2gL(r_1 + 0.5g)\theta$. Consequently, the coenergy is equal to the product of the air gap coenergy density and the air gap volume, i.e.,

$$W_g' = \left(\frac{\mu_0 H_g^2}{2}\right)[2gL(r_1 + 0.5g)\theta) = \frac{\mu_0 (Ni)^2 L(r_1 + 0.5g)\theta}{4g}$$

and the torque is given by eqn (1.43), i.e.,

$$T_{elm} = \frac{\partial W_g'(i,\theta)}{\partial \theta}|_i = \frac{\mu_0 (Ni)^2 L(r_1 + 0.5g)}{4g}$$

The sign of the torque is positive, hence acting in the direction to increase the overlap angle θ and thus to align the rotor with the stator poles.

(b) Calculate the maximum torque for $B_g = 1.6$ T, $r_1 = 28$ mm, $L = 20$ mm and $g = 1.5$ mm

$$H_g = \frac{B_g}{\mu_0} = \frac{1.6}{04\pi \times 10^{-6}} = 1.273 \times 10^6 \text{ A/m}$$

Thus, the MMF

$$Ni = 2gH_g = 2 \times 0.0015 \times 1.31 \times 10^{-6} = 3820 \text{ A}$$

The torque

$$T_{elm} = \frac{0.4\pi 10^{-6} \times 8000^2 \times 0.02(0.028 + 0.5 \times 0.0015)}{4 \times 0.0015} = 0.217 \text{ Nm}$$

Example 1.3

In the doubly-excited device shown in Fig. 1.20, the inductances are $L_{11} = 2.0 + 2.0\cos(2\theta) \times 10^{-3}$ H, axial length (perpendicular to the page) $L_{12} = 0.5\cos(\theta)$ H, $L_{22} = 25.0 + 15.0\cos(2\theta)$ H. Find the torque $T(\theta)$ for current $i_1 = 1.2$ A and $i_2 = 0.04$ A. Calculate the torque for the above values of currents and $\theta = 10°$, $\theta = 60°$ and $\theta = 120°$.

Solution

$$\frac{dL_{11}(\theta)}{d\theta} = \frac{2.0 + 2.0\cos(2\theta) \times 10^{-3}}{d\theta} = -4.0\sin(2\theta) \times 10^{-3}$$

$$\frac{dL_{22}(\theta)}{d\theta} = \frac{25.0 + 15.0\cos(2\theta)}{d\theta} = -30\sin(2\theta)$$

$$\frac{dL_{12}(\theta)}{d\theta} = \frac{0.5\cos(\theta)}{d\theta} = -0.5\sin(\theta)$$

The electromagnetic torque is given by eqn (1.43)

$$T_{elm} = \frac{i_1^2}{2}\frac{dL_{11}(\theta)}{d\theta} + \frac{i_2^2}{2}\frac{dL_{22}(\theta)}{d\theta} + i_1 i_2 \frac{dL_{12}(\theta)}{d\theta}$$

$$= \frac{1.2^2}{2}[-4.0\sin(2\theta) \times 10^{-3}] + \frac{0.4^2}{2}[-30\sin(2\theta)] - 1.2 \times 0.04[-0.5\sin(\theta)]$$

$$= -1.4029\sin(2\theta) - 0.24\sin(\theta)$$

For $\theta = 10°$ the torque $T = -0.013$ Nm, for $\theta = 60°$ the torque $T = -0.044$ Nm and for $\theta = 120°$ the torque $T = 2.494 \times 10^-3$ Nm.

Notice that the torque expression consists of terms of two types. One term, proportional to $i_1 i_2 \sin(\theta)$, is due to the mutual interaction between the rotor and stator currents. It acts in a direction to align the rotor and stator so as to maximize their mutual inductance.

The torque expression also has two terms each proportional to $\sin(2\theta)$ and to the square of one of the coil currents. These terms are due to the action of the individual winding currents alone and correspond to the torques one sees in singly-excited systems. Here the torque is due to the fact that the self inductances are a function of rotor position and the corresponding torque acts in a direction to maximize each inductance so as to maximize the coenergy. The 2θ variation is due to the corresponding variation of the self indutances, which in turn is due to the variation of the air gap inductance.

1.7 Basic coordinates and parameters of systems

The following forms of energy occur in the process of energy conversion in *electromechanical systems* (e.g., in electrical machines):

(a) *External energy* absorbed or produced by the machine;
(b) *Energy stored* in the machine or elements combined with the machine;
(c) *Dissipated energy*, e.g., in the form of heat.

Basic coordinates of processes of conversion can relate to electrical and mechanical parts. Basic coordinates of electric systems can be found under assumption, that all systems consist of *concentrated-parameter elements*, *conservative elements* (in which the energy is stored) and *dissipative elements* (in which the energy is dissipated).

Capacitive elements and *inductive elements* are conservative elements. *Resistive elements* are dissipative elements. The theory of electromagnetic field deals with *distributed-parameter elements and systems*.

1.7.1 Capacitive element

The basic coordinate that describes a *capacitive element* C is the electric charge q, because its quantity stored in this element determines a measure of the energy stored. The equation describing a linear capacitive element has the following form:

$$v(t) = \frac{1}{C} q(t) \tag{1.46}$$

When the capacitance C is a nonlinear function of the voltage $v(t)$, the function or their variability (variation) must be known in order to describe the state of the element. If the characteristic $q(t)$ is a straight line, the characteristic $v(t)$ in the case of linear capacitance C is a straight line too.

The electric current in the capacitive element is the first derivative of the electric charge q with respect to the time t. This quantity (electric current i) can be easily measured and is commonly used in electrical engineering, i.e.,

$$\frac{dq}{dt} = \dot{q} = i \tag{1.47}$$

1.7.2 Inductive element

Similar to capacitive element, the *inductive element* L can also store the energy. The basic coordinate that describes this element is coupled magnetic flux ψ (linkage flux). The equation describing a linear inductive element has the form:

$$i(t) = \frac{1}{L} \psi(t) \tag{1.48}$$

When the inductance L is a nonlinear function of the electric current $i(t)$ (element with a ferromagnetic core), the function or its graph must be known to describe the state of the element. The electric voltage v is the first derivative of the magnetic flux ψ with respect to the time t, i.e.,

$$\frac{d\psi}{dt} = \dot{\psi} = v \tag{1.49}$$

Similar to the current i, the voltage v can be easily measured.

1.7.3 Resisitive element

A *resistive element* R is a dissipative element. It can be described using q and Ψ as basic coordinates.

The following relationship exists for a concentrated dissipative element with its conductance G

$$\dot{q} = i = G\dot{\psi} = \frac{1}{R} \tag{1.50}$$

where the current i is according to eqn (1.47) and the electric induced voltage v is according to eqn (1.49). The element is linear if the conductance G of the medium is constant. The element is nonlinear if the conductance G is a function of current or voltage. In this case the conductance $G(\dot{\psi})$ can be found on the basis of the characteristic $\dot{q} = G(\dot{\psi})$ for a given voltage $\dot{\psi}$.

1.7.4 Mass in translatory motion

The momentum p is the basic coordinate of the *mass* m in translatory motion, because the momentum is a measure of the energy stored by an element of mass. The equation describing a *linear element of mass* has the following form:

$$v(t) = \frac{1}{m}p(t) \tag{1.51}$$

If the mass m was dependent on the velocity v (as it is in relativistic physics) or dependent on the momentum p, then the characteristic of the element of mass $p = f(v)$ would be curvilinear. For a given velocity v the point corresponding to the velocity on this curve would describe the mass of the element. The first derivative of the momentum with respect to time

$$\frac{dp}{dt} = \dot{p} = f \tag{1.52}$$

is equal to the force f acting on the element of mass.

1.7.5 Elastic element in translatory motion

In most cases the *elastic element* is described by the x coordinate. The equation describing this element has the form

$$f(t) = \frac{1}{K}x(t) \tag{1.53}$$

where f is the force acting on the spring, N, K is the *compliance* of the elastic element, m/N. The inverse of compliance is the *stiffness* of the spring, N/m. Stiffness is the resistance of an elastic body to deformation by an applied force.

For full description of a nonlinear elastic element, the characteristic $x = f(F)$ is required. This characteristic defines the compliance in each point of the element and for each state of the element.

The momentum p and position x are the basic mechanical coordinates in translatory motion. Their first derivatives with respect to time are force and velocity, i.e.,

$$f = \frac{dp}{dt} = \dot{p} = \frac{d}{dt}(mv) \tag{1.54}$$

and velocity

$$v = \frac{dx}{dt} = \dot{x} \tag{1.55}$$

1.7.6 Dissipative element in translatory motion

For a clear description of a *dissipative element* D (viscous damping), basic coordinates for translatory motion are sufficient because a linear damping element is described by the following equation:

$$\dot{p} = D\dot{x} \tag{1.56}$$

where D is the damping coefficient, N/(m/s), which is numerically equal to the slope of the characteristic of the element $\dot{p} = D\dot{x}$. In general, this element can also be nonlinear, similar to all other elements previously described.

1.7.7 Concentrated-parameter elements in rotary motion

All *concentrated-parameter elements* in rotary motion, i.e.,

- Inertia J,
- Torsional compliance K_ϑ,
- Torsional damping D_ϑ

can be defined in a similar way as in translatory motion. In the case of rotary motion, the basic coordinates are

- Angular momentum

$$l = mr^2 \Omega(t) = J\Omega(t) \tag{1.57}$$

- Angular position ϑ.

Their first derivatives with respect to the time t are

- Torque

$$T = \frac{dl}{dt} = \dot{l} \tag{1.58}$$

- Angular velocity

$$\Omega = \frac{d\vartheta}{dt} = \dot{\vartheta} \tag{1.59}$$

The moment of inertia of a material point is

$$J = mr^2 \tag{1.60}$$

while the *centrifugal force*

$$F_r = mr\dot{\vartheta}^2 = mr\Omega^2 \tag{1.61}$$

Equations describing concentrated elements in rotary motion are as follows:

- Inertial element

$$\Omega(t) = \frac{1}{J} l(t) \tag{1.62}$$

- Elastic element

$$T(t) = \frac{1}{K_\vartheta} \vartheta(t) \tag{1.63}$$

- Torsional damping element

$$\dot{l} = D_\vartheta \dot{\vartheta} \tag{1.64}$$

All basic coordinates of energy conversion, their first derivatives and parameters describing each element are set up in Table 1.4.

Table 1.4. Basic coordinates of energy conversion, their first derivatives and parameters describing each element.

Form of energy	Mechanical translatory motion	Mechanical rotary motion	Electrical
Coordinates	Momentum, p, Linear position, x	Angular momentum, l, Angular position, ϑ	Electric charge, q Linkage flux, ψ
Their first derivative with respect to time	Force, $f = \dot{p}$ Velocity, $v = \dot{x}$	Torque, $T = \dot{l}$ Angular velocity, $\Omega = \dot{\vartheta}$	Electric current, $i = \dot{q}$ Electric voltage, $v = \dot{\psi}$
Mathematical description	Translatory motion	Rotary motion	Electric system
Conservative element	Mass $m(\dot{x}) = dp/d\dot{x}$	Moment of inertia $J(\dot{\vartheta}) = dl/d\dot{\vartheta}$	Capacitance $C(\dot{\psi}) = dq/d\dot{\psi}$
Conservative element	Compliance $K(\dot{p}) = dx/d\dot{x}$	Torsional compliance $K(\dot{\vartheta}) = d\vartheta/d\dot{l}$	Inductance $L(\dot{q}) = d\psi/d\dot{q}$
Dissipative element	Damping (friction) $D(\dot{x}) = d\dot{p}/d\dot{x}$	Torsional damping $D_{\vartheta}(\dot{\vartheta}) = d\dot{l}/d\dot{\vartheta}$	Conductance $G(\dot{\psi}) = d\dot{q}/d\dot{\psi}$

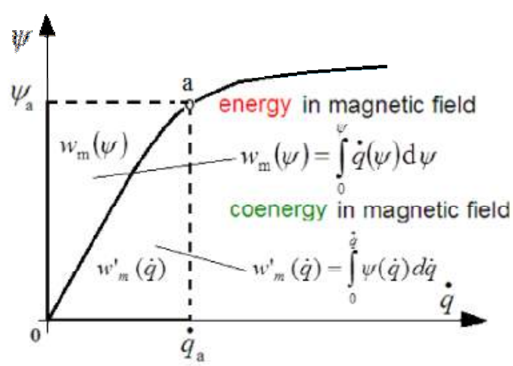

Fig. 1.23. Magnetic flux–current characteristic of a nonlinear inductive element.

1.8 Energy and coenergy

1.8.1 Energy and coenergy of a nonlinear inductive element

If basic coordinates and their first derivatives are known, equations describing energies in the process of conversion can be obtained. A nonlinear inductive element with flux–current characteristic sketched in Fig. 1.23 will be considered.

Coenergy (a second state function of the energy) is an auxiliary function necessary for calculations of the force or torque at constant current. In the linear case, energy and coenergy are numerically equal.

The instantaneous value of the power in a single coil is

$$P(t) = \dot{q}\dot{\psi} \tag{1.65}$$

The energy influx per unit of time to the inductive element is the energy growth stored in the magnetic field and described by the formula

$$dW_m(t) = P(t)dt = \dot{q}\frac{d\psi}{dt}dt \tag{1.66}$$

The total energy delivered in the time interval $t \in (t_0, t)$ is calculated as an integral of eqn (1.66), i.e.,

$$W_m(t) - W_m(t_0) = \int_{t_0}^{t} \dot{q}\frac{d\psi}{dt}dt \tag{1.67}$$

or in equivalent form

$$W_m(\psi) - W_m(\psi_0) = \int_{\psi_0}^{\psi} \dot{q}(\psi)d\psi \tag{1.68}$$

Assuming that at the time instant t_0 the linkage flux $\psi = 0$, eqn (1.68) can be simplified to the form

$$W_m(\psi) = \int_{0}^{\psi} \dot{q}(\psi)d\psi \tag{1.69}$$

Eqn (1.69) represents the total energy of the magnetic field stored in an inductive element. This is the surface area above the curve $\psi(\dot{q})$ in Fig. 1.23. In the case of linear element this energy is expressed by the formula

$$W_m = \frac{1}{2}\dot{q}\psi = \frac{1}{2}i\psi \tag{1.70}$$

Putting $\psi = Li$ to eqn (1.70), the total energy of the magnetic field is

$$W_m = \frac{1}{2}Li^2 \tag{1.71}$$

This is the well-known energy formula in the theory of electric circuits.

The area below the curve (Fig. 1.23) described by the equation

$$W'_m(\dot{q}) = \int_{0}^{\dot{q}} \psi(\dot{q})d\dot{q} = \tag{1.72}$$

is called the *coenergy* (in this case the magnetic coenergy). The following relationship exists between the energy and coenergy:

$$W_m + W'_m = \dot{q}\psi \tag{1.73}$$

The last eqn (1.73) says that the stored energy is equal to cooenergy (Fig. 1.23). This relationship is true both for linear and nonlinear elements. In the case of linear elements, the energy W_m is equal to the coenergy W'_m.

1.8.2 Energy and coenergy of a nonlinear capacitive element

In a similar way relationships describing the energy and coenergy in a capacitive element can be derived, i.e.,

- Electrical energy

$$W_e(q) = \int_0^q \dot{\psi}(q)dq \tag{1.74}$$

- Electrical coenergy

$$W'_e(\dot{\psi}) = \int_0^{\dot{\psi}} q(\dot{\psi})d\dot{\psi} \tag{1.75}$$

In the case of a linear element

$$W_e = \frac{1}{2}\dot{\psi}q = \frac{1}{2}vq = \frac{1}{2}Cv^2 = W'_e \tag{1.76}$$

This is the triangle area below the straight line $v = \dot{\psi} = f(\dot{q})$.

1.8.3 Energy and coenergy of mechanical systems

For mechanical systems with translatory or rotary motion the kinetic energy and coenergy can be found in a similar way. Appropriate relationships for instantaneous powers delivered to mass or inertia elements have the following forms:

- For translatory motion

$$P(t) = \dot{x}(t)\dot{p}(t) = v(t)F(t) \tag{1.77}$$

- For rotary motion

$$P(t) = \dot{\vartheta}(t)\dot{l}(t) = \Omega(t)T(t) \tag{1.78}$$

Kinetic energy and kinetic coenergy in translatory motion are described by the following equations (Fig. 1.24a):

$$W_k(p) = \int_0^p \dot{x}(p)dp \qquad \text{and} \qquad W'_k(\dot{x}) = \int_0^{\dot{x}} p(\dot{x})d\dot{x} \tag{1.79}$$

In rotary motion it is necessary to replace the linear velocity $\dot{x} = v$ with the angular velocity $\dot{\vartheta} = \Omega$ and the force $\dot{p}$ with the torque $\dot{l} = T$. Potential energy and potential coenergy, which describe the state of elastic elements, per analogy, are expressed by the following equations (Fig. 1.24b):

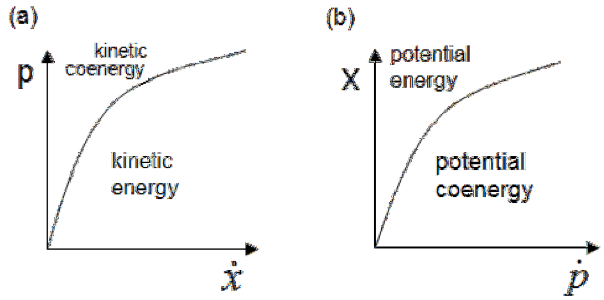

Fig. 1.24. Energy and coenergy in mechanical systems: (a) kinetic energy and coenergy; (b) potential energy and coenergy.

$$W_p(x) = \int_0^x \dot{p}(x)dx \qquad \text{and} \qquad W_p'(\dot{p}) = \int_0^{\dot{p}} x(\dot{p})d\dot{p} \tag{1.80}$$

In this case, similar to conservative electrical elements, the following relationships result (Fig. 1.24):

$$W_k + W_k' = p\dot{x} \qquad \text{and} \qquad W_p + W_p' = x\dot{p} \tag{1.81}$$

To determine the state function of a system consisting of multiple elements, it is necessary to add the adequate energies. Then, eqns (1.66) to (1.80) will become functions of many coordinates or generalized velocities. Tables 1.5 and 1.6 contain all relationships describing the energy stored in linear conservative elements and discharged in dissipative elements.

Example 1.4.

Find the kinetic energy and momentum of

(a) Passenger aicraft with mass of 78.6 t that lands with speed of 270 km/h;
(b) Passenger car with mass 1.5 t moving with velocity of 100 km/h.

(a) Landing passenger aicraft

$$W_k = \frac{1}{2}mv^2 = \frac{1}{2}78,600\left(\frac{270}{3.6}\right)^2 = 221,062,500 \text{ J} = 221.06 \text{ MJ}$$

$$p = mv = 78,600\left(\frac{270}{3.6}\right) = 5,895,000 \text{ Ns} = 5895 \text{ MNs}$$

Table 1.5. Energy stored and dissipated in linear concentrated-parameter electrical elements.

Elements	Electrical elements	
	Parameter	Energy
Conservative	Capacitance	
	$C = \frac{dq}{d\psi}$	$w_e = \frac{1}{2}Cv^2 = \frac{1}{2}C(\dot{\psi})^2$
	Inductance	
	$L = \frac{d\psi}{d\dot{q}}$	$w_m = \frac{1}{2}Li^2 = \frac{1}{2}L(\dot{q})^2$
Dissipative	Resistance	
	$L = \frac{d\dot{\psi}}{d\dot{q}}$	$\frac{dw_R}{dt} = Ri^2 = R(\dot{q})^2$
	Conductance	
	$G(\dot{\psi}) = \frac{d\dot{q}}{d\dot{\psi}}$	$\frac{dw_R}{dt} = Gv^2 = G(\dot{\psi})^2$

Table 1.6. Energy stored and dissipated in linear concentrated-parameter mechanical elements.

Elements	Mechanical elements			
	Translatory motion		Rotary motion	
	Parameter	Energy	Parameter	Energy
Conservative	Mass		Inertia	
	$m(\dot{x}) = \frac{dp}{d\dot{x}}$	$w_k = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{x})^2$	$J(\dot{\vartheta} = \frac{dl}{d\dot{\vartheta}}$	$w_k = \frac{1}{2}J\Omega^2 = \frac{1}{2}J(\dot{\vartheta})^2$
	Compliance		Torsional compliance	
	$K(\dot{p}) = \frac{dx}{d\dot{p}}$	$w_p = \frac{1}{2}KF^2 = \frac{1}{2}K(\dot{p})^2$	$K_\vartheta(\dot{l}) = \frac{d\vartheta}{d\dot{l}}$	$w_p = \frac{1}{2}K_\vartheta T^2 = \frac{1}{2}K_\vartheta(\dot{l})^2$
Dissipative	Damping (friction)		Torsional friction	
	$D(\dot{x}) = \frac{d\dot{p}}{d\dot{x}}$	$\frac{dw_R}{dt} = Dv^2 = D(\dot{x})^2$	$D_\vartheta(\dot{\vartheta} = \frac{d\dot{l}}{d\dot{\vartheta}}$	$\frac{dw_R}{dt} = D_\vartheta\Omega^2 = D_\vartheta(\dot{\vartheta})^2$

(a) Passenger car

$$W_k = \frac{1}{2}mv^2 = \frac{1}{2}1500\left(\frac{100}{3.6}\right)^2 = 1,157,407 \text{ J} = 1.16 \text{ MJ}$$

$$p = mv = 1500\left(\frac{100}{3.6}\right) = 41,667 \text{ Ns} \approx 0.042 \text{ MNs}$$

The consequences of collison with an obstacle, e.g., tree or wall, will be much more serious for the passenger car because its kinetic energy and momentum are much smaller than those of a heavy aircraft.

1.9 Force and torque balance equations

For a system of n particles, d'Alembert principle can be expressed as follows:

The work performed by the sum of external forces and inertia forces on the distance that is a virtual displacement, or virtual work, is equal to zero. This principle can be expressed with the aid of the following equation:

$$\sum_{i=1}^{n} (\mathbf{F}_i + \mathbf{F}_{Ji})\, \delta \mathbf{r}_i \tag{1.82}$$

where $\mathbf{F}_i$ is the total applied force (excluding constraint forces) acting on the ith particle of the system, $\mathbf{F}_{Ji} = -m_i \mathbf{a}_i$ is the force of inertia acting on the ith particle with its mass m_i, $\mathbf{a}_i$ is the accelleration of the ith particle of the system, and $\delta \mathbf{r}_i$ is the virtual displacement of the ith particle of the system. The product $m_i \mathbf{a}_i$ represents the time derivative of the momentum of the ith particle

On the basis of the d'Alembert principle for translatory motion

$$m\frac{dv}{dt} + Dv + K\int v dt = F \tag{1.83}$$

where F is the sum of external and electromagnetic forces. Eqn (1.83) can also be written in the form

$$m\frac{d^2x}{dt^2} + D\frac{dx}{dt} + Kx = F \tag{1.84}$$

or

$$m\ddot{x} + D\dot{x} + Kx = F \tag{1.85}$$

Similar equations can be written for a series electrical circuit RLC

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = v \tag{1.86}$$

or

$$L\ddot{q} + R\dot{q} + \frac{1}{C}q = v \tag{1.87}$$

For rotary motion the torque balance equation is

$$J\frac{d^2\vartheta}{dt^2} + D_\vartheta\frac{d\vartheta}{dt} + K_\vartheta\vartheta = T \tag{1.88}$$

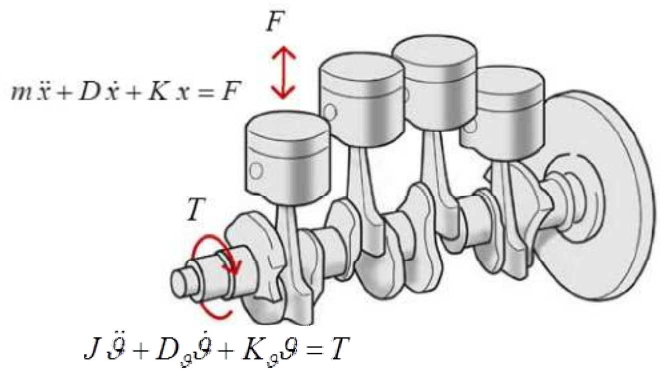

Fig. 1.25. Crankshaft: conversion of reciprocating motion $m\ddot{x} + D\dot{x} + Kx = F$ into rotary motion $J\ddot{\vartheta} + D_\vartheta\dot{\vartheta} + K_\vartheta\vartheta = T$.

or

$$J\ddot{\vartheta} + D_\vartheta\dot{\vartheta} + K_\vartheta\vartheta = T \tag{1.89}$$

where T is the sum of external and electromagnetic torques, D_ϑ is the torsional damping, K_ϑ is the torsional compliance and T_ϑ is the external torque.

Fig. 1.25 shows a crankshaft of a combustion engine. A crankshaft is a basic part of combustion engine able to perform a conversion of reciprocating motion of a piston into rotational motion of shaft. The reciprocating motion of pistons can be described by eqns (1.83), (1.84), and (1.85), while the rotary motion of the shaft can be described by eqns (1.88) and (1.89).

Summary

Electromechanical energy conversion is a conversion of mechanical energy into electrical energy (generator) or vice versa (motor) with the aid of rotary motion (rotary machines) or translatory (linear, reciprocating) motion (linear machines and actuators).

Electrical machines, solenoid actuators and electromagnets are generally called *electromechanical energy conversion devices*.

Transformers and solid-state converters do not belong to the group of electromechanical energy conversion devices because they only convert one kind of electrical energy into another kind of electrical energy with different parameters (change in voltage, current, frequency, number of phases, conversion of DC into AC current, etc.) without any motion.

The corresponding electric circuit parameters and magnetic circuit parameters are: electric voltage, V – magnetic voltage drop, V_μ; SEM, E – MMF,

F; electric current, I – magnetic flux, Φ; resistance, R – reluctance, R_μ; conductance, $G = 1/R$ – permeance, Λ_μ; electric conductivity, σ – magnetic permeability, μ. Other analogies are given in Table 1.2.

Ferromagnetic materials for cores of electrical machines and electromagnetic devices are magnetically nonlinear (nonlinear variation of magnetic flux density B with magnetic field intensity H) and are described by magnetization curves $B(H)$ and specific core loss curves $\Delta p(B)$.

Electromagnetic torque of a rotary doubly-excited electromechanical energy conversion device (Fig. 1.20) has two components: synchronous torque $T_{ms}\sin(\vartheta)$ and reluctance torque $T_{mrel}\sin(2\vartheta)$, i.e.,

$$T_{elm} = -i_1 i_2 L_0 \sin(\vartheta) - (i_1^2 \Delta L_1 + i_2^2 \Delta L_2)\sin(2\vartheta)$$

$$= -\left[T_{ms}\sin(\vartheta) + T_{mrel}\sin(2\vartheta)\right]$$

All elements in electrical and mechanical systems can be described by the following equations:

- Capacitive element, inductive element and resistive element

$$v(t) = \frac{1}{C}q(t) \qquad i(t) = \frac{1}{L}\psi(t) \qquad \dot{q} = G\dot{\psi} = \frac{1}{R}\dot{\psi}$$

- Element of mass, elastic element and dissipative element (viscous friction) in translatory motion

$$v(t) = \frac{1}{m}p(t) \qquad F(t) = \frac{1}{K}x(t) \qquad \dot{p} = D\dot{x}$$

- Inertial element, elastic element and torsional damping element in rotary motion

$$\Omega(t) = \frac{1}{J}l(t) \qquad T(t) = \frac{1}{K_\vartheta}\vartheta(t) \qquad \dot{l} = D_\vartheta\dot{\vartheta}$$

Characteristic $\psi(i)$ of a nonlinear inductive element is plotted in Fig. 1.23. The area above the curve is the energy stored in magnetic field

$$W_m(\psi) = \int_0^{\psi} \dot{q}(\psi)d\psi$$

The area below the curve is the so-called *coenergy*, i.e.,

$$W'_m(\dot{q}) = \int_0^{\dot{q}} \psi(\dot{q})d\dot{q}$$

Coenergy (a second state function of the energy) is an auxiliary function necessary for calculations of the force or torque at constant current. The sum of energy and coenergy, both for linear and nonlinear elements, is equal to the area of the rectangle $\dot{q}\psi$, or

$$W_m + W_m' = \dot{q}\psi$$

In the case of a linear element, the energy W_m is equal to coenergy W_m', i.e.,

$$W_m = \frac{1}{2}\dot{q}\psi = Wm' = \frac{1}{2}\psi\dot{q}$$

For a capacitive element, the energy stored in electric field and coenergy are, respectively,

$$W_e(q) = \int_0^q \dot{\psi}(q)dq \qquad\qquad W_e'(\dot{\psi}) = \int_0^{\dot{\psi}} q(\dot{\psi})d\dot{\psi}$$

In the case of a linear capacitive element

$$W_e = \frac{1}{2}\dot{\psi}q = W_e' = \frac{1}{2}q\dot{\psi}$$

Energy and coenergy in mechanical motion have the following forms:

- For translatory motion

$$W_k(p) = \int_0^p \dot{x}(p)dp \qquad\qquad W_k'(\dot{x}) = \int_0^{\dot{x}} p(\dot{x})d\dot{x}$$

- For rotary motion

$$W_k(l) = \int_0^l \dot{\vartheta}(l)dl \qquad\qquad W_k'(\dot{\vartheta}) = \int_0^{\dot{\vartheta}} l(\dot{\vartheta})d\dot{\vartheta}$$

Energies and coenergies are functions of generalized velocity, i.e., first derivatives with respect to time of basic coordinates, i.e., electric current, $i = \dot{q}$, electric voltage, $v = \dot{\psi}$, linear velocity, $v = \dot{x}$, momentum, p, angular velocity, $\varOmega = \dot{\vartheta}$ and angular momentum, l.

On the basis of d'Alembert's principle, the force balance equation in translatory motion has the form

$$m\ddot{x} + D\dot{x} + Kx = F$$

A similar equation for torque balance in rotary motion:

$$J\ddot{\vartheta} + D_\vartheta\dot{\vartheta} + K_\vartheta\vartheta = T$$

Problems

1. A ferromagnetic ring made of M19 laminated silicon steel has a circular cross-section with diameter $d = 0.02$ m and mean diameter $D = 0.3$ m. The ring is uniformly wound with a coil of $N = 1000$ turns. The magnetization curve of M19 silicon steel is plotted in Fig. 1.8.

   (a) Find the current in the coil required to produce the magnetic flux of $\Phi = 0.0004$ Wb in the ring;
   (b) If an air gap $g = 0.002$ m is cut in the ring, find the air gap flux produced by the current found in (a);
   (c) Find the current that produces the same flux in the air gap as in (a).

   Answer: (a) $I = 0.304$ A; (b) $\Phi_g = 5.216 \times 10^{-5}$ Wb; (c) $I' = 2.33$ A.

2. A coil with the number of turns $N = 520$ and resistance $R = 12.0\ \Omega$ is fed with DC voltage $V_{dc} = 3.0$ V. The coil is wound uniformly on a ring-shaped core with rectangular cross-section made of silicon steel tape M19 (Fig. 1.8). The inner diameter of the core $D_{in} = 0.15$ m, outer diameter $D_{out} = 0.2$ m, and its thickness $L = 0.08$ m. Find the magnetic flux density B_c and magnetic flux Φ in the core. The stacking factor $k_i = 0.96$.

   Answer: $B_c = 1.23$ T, $\Phi = 0.00236$ Wb.

3. The ring-shaped core with coil of Problem 1.2 is fed with AC voltage $V = 380$ V and frequency $f = 50$ Hz. The specific mass density of M19 silicon steel is $\rho_{Fe} = 7650$ W/kg. Find: (a) the magnetizing current I_Φ; (b) core losses ΔP_c; (c) total current I_0 drawn by the coil.
   The voltage drop across the coil resistance, additional losses and excess eddy current losses can be neglected.

   Answer: (a) $I_\Phi = 6.53$ A; (b) $\Delta P_c = 30.9$ W; (d) $I_0 = 6.53$ A.

4. For the ferromagnetic core with coil of Problems 1.2 and 1.3 calculate the self-inductance, if the coil is fed with DC voltage: (a) $V_1 = 1.0$ V; (b) $V_2 = 6.0$ V.
   For each voltage find the relative magnetic permeablity using the $B--H$ curve plotted in Fig. 1.8.

   Answer: (a) $\mu_{r1} = 8047$, $L_1 = 9.55$ H; (b) $\mu_{r2} = 1341$, $L_2 = 1.59$ H.

5. A simple electrical machine has the following dimensions of ferromagnetic core:
   - Stator core outer diameter $D_{1out} = 0.16$ m;
   - Stator core inner diameter $D_{1in} = 0.11$ m;
   - Rotor core outer diameter $D_{2out} = 0.108$ m;
   - Rotor core inner diameter $D_{2in} = 0.055$ m;

- Axial length of core $L = 0.1$ m.

The relative magnetic permeability of the stator and rotor core is $\mu_r = 500$ and the magnetic flux density in the air gap is $B_g = 1.0$ T. Find:
(a) The ferromagnetic core-to-air gap volume ratio k_V;
(b) The ferromagnetic core-to-air gap energy ratio k_W.

Answer: (a) $k_V = 50.78$; (b) $k_W = 0102$.

6. An electromagnetic relay has an exciting coil of $N = 1000$ turns. The coil has a cross-sectional area $a \times b = 50 \times 60$ mm^2. The reluctance of the magnetic circuit and fringing effect are neglected. Find:
(a) The coil inductance L_g if the air gap is $g = 3$ mm;
(b) The stored magnetic field energy W_m for a coil current of $i = 1.5$ A;
(c) The mechanical energy output based on field energy changes dW_m when the armature moves to position for which $g' = 1.5$ mm and coil current remains constant at $i = 1.5$ A. Assume slow movement of armature;
(d) Repeat (c) above based on force-calculations and mechanical displacement.

Answer: (a) $L_g = 1.257$ H; (b) $W_m = 1.414$ J; (c) $dW_m = 1.414$ J; (d) $W_{mech} = 1.414$ J.

7. The magnetic circuit shown in Fig. 1.20 is made of high-permeability electrical steel. The rotor does not have any winding and is free to turn about a vertical axis. The leakage flux and fringing effect are neglected.
(a) Derive an expression for the torque acting on the rotor in terms of the self-inductance L_s;
(b) Calculate the torque for the radius of the rotor $r = 28$ mm, axial length (perpendicular to the page) of the magnetic circuit $L = 80$ mm, number of the stator turns $N = 1100$, stator current $i = 2.0$ A and the air gap between the rotor and stator $g = 1.5$ mm;
(c) The maximum flux density in the overlaping portions of the air gaps is to be limited to approximately 1.6 T to avoid excessive saturation of the steel. Calculate the maximum current and torque for the dimensions and numbers of turns as above.

Answer: (a) $L_s = \mu_0 L(r{+}0.5g)N^2\theta/(2g)$, $T_{elm} = \mu_0 L(r{+}0.5g)(iN)^2/(4g)$; (b) $T_{elm} = 3.467$ Nm; (c) $i_{max} = 2.315$ A, $T_{elmmax} = 4.645$ Nm.

8. In the doubly-excited device shown in Fig. 1.20, the inductances are $L_{11} = 2.0 + 2.0\cos(2\theta) \times 10^{-3}$ H, $L_{12} = 0.5\cos(\theta)$ H, $L_{22} = 25.0 + 15.0\cos(2\theta)$ H. Find and plot the torque $T(\theta)$ for current $i_1 = 1.2$ A and $i_2 = 0.04$ A. Calculate the torque for the above values of currents and $\theta = 10°$, $\theta = 60°$ and $\theta = 120°$.

Answer: For $\theta = 10°$ $T = -0.013$ Nm; for $\theta = 60°$ $T = -0.044$ Nm and for $\theta = 120°$ $T = 2.494 \times 10^{-}3$ Nm.

9. Find the stored magnetic field energy and the associated power in a coil of inductance $L = 0.003$ H after $t = 0.001$ s, if the coil is fed with sinusoidal voltage $v = V_m \sin(\omega t)$, $V_m = 311.1$ V, $f = 50$ Hz.

   Answer: $W_m = 147.8$ J, $P = -30\ 180$ W.

10. An AC voltage $v = V_m \sin(\omega t)$ is applied to the capacitor $C = 0.000020$ F. The frequency is $f = 50$ Hz, and the rms voltage is $V = 100$ kV. Find the stored electric field energy after $t = 0.001$ s and associated maximum power.

    Answer: $W_e = 19\ 100$ J, $P_{max} = 62.83 \times 10^6$ W.

11. A ferromagnetic UI-core with two air gaps as shown in Fig. 2.3b has the following dimensions: width of leg $a = 0.05$ m, height of yoke $a = 0.05$ m, height of U-core $h = 0.25$ m, width of U-core (equal to the width of I-shaped moving armature) $b = 0.2$ m, thickness of stack $L = 0.07$ m. The core is made of a silicon electrical steel M19 (Fig. 1.8). The stacking factor is $k_i = 0.96$ and the specific mass density of the silicon steel is $\rho_F e = 7650$ kg/m^3. The air gap depends on the position of moving armature (I-shaped). The number of turns would on stationary U-section is $N = 200$. The winding is fed with AC voltage $V = 127$ V amd frequency $f = 50$ Hz. Calculate:
    (a) The magnetizing current for air gaps $g_0 = 0$, $g_1 = 0.001$ m and $g_2 = 0.002$ m;
    (b) Total current including core loss current for the same air gaps;
    (c) Attraction force between I-shaped armature and stationary U-shaped core and energy stored in magnetic field for the same air gaps.

    Answer: (a) $I_{\Phi 0} = 0.229$ A for $g_0 = 0$, $I_{\Phi 1} = 5.016$ A for $g_0 = 1$ mm, $I_{\Phi 2} = 9.804$ A for $g_0 = 2$ mm; (b) the core loss current is only $I_{Fe} = 0.006$ A so the total currents are the same as magnetizing currents; (c) $F_0 \to \infty$ for $g_0 = 0$ if magnetic saturation is not included, $W_{m0} = 0.046$ J, $F_1 = 1062.5$ N, $W_{m1} = 1.063$ J for $g_1 = 1.0$ mm, $F_2 = 1015.0$ N $W_{m2} = 2.029$ J for $g_2 = 2.0$ mm.

12. The UI-core with two air gaps of Problem 1.11 as shown in Fig. 2.3b is fed with DC voltage $V_{dc} = 6.0$ V. The winding is wound with the same number of turns $N = 200$ using a round wire with diameter of bare wire $d_w = 1.0236$ mm (AWG 18). The thickness of core-to-coil insulation is $t_i = 1.0$ mm. Find the attractive force and stored magnetic energy for air gaps $g_1 = 1.0$ mm and $g_2 = 2.0$ mm. The electric conductivity of copper at 75°C is $\rho_{75} = 47 \times 10^6$ S/m/ The mean length of turn can be approximately calculate from the formula $l_{mean} = 2(a + L + 2t_i + 2d_w)$.

Answer: $F_1 = 923.5$ N, $W_{m1} = 0.924$ J at $g_1 = 1.0$ mm; $F_2 = 230.9$ N, $W_{m2} = 0.462$ J at $g_1 = 2.0$ mm.

13. A disk with its diameter $D = 1.5$ m, thickness $t_d = 0.16$ m and specific mass density $\rho_d = 7700$ kg/m^3 rotates with the speed of $n = 480$ rpm. Calculate: (a) moment of inertia J; (b) stored kinetic energy E_k; (c) input external torque, to obtain the rotational speed of $n = 480$ rpm in the time $t = 15$ s.

    Answer: (a) $J = 612.3$ kgm^2; (b) $E_k = 773.5 \times 10^3$ J; (c) $T = 2052$ Nm.